

On time to buffer saturation in a $GI/M/1/N$ -type queue

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Abstract—A $GI/M/1/N$ -type queuing system with independent and generally distributed interarrival times and exponential service times is investigated. A system of equations for conditional distributions of the time to the first buffer saturation is built. The solution is written using a special-type sequence defined by “input” distributions of the system. The formula of total probability is used to derive a representation for the distribution of the time to the k th buffer saturation for $k \geq 2$. Moreover, special cases of the Poisson arrival process and the system with one-place buffer are discussed. Sample numerical results for the 3-Erlang and deterministic distributions of interarrival times are attached as well.

Keywords—Buffer saturation, finite-buffer queue, formula of total probability, integral equation, Markov moment.

I. INTRODUCTION

A phenomenon of buffer saturation and, in consequence, losses of the arriving packets is a typical one in telecommunication networks. As one can note, the well-known performance measure as the loss ratio, defined as a part of the total amount of packets in transmission that is lost due to the buffer saturation, does not give a sufficient information about the process of losses from the probabilistic point of view. The in-depth analysis of the problem of losses requires the knowledge about distributions of times to successive buffer saturations (or, in other words, periods during which the service process is not blocked) and durations of such periods.

The review of results for stochastic characteristics of queuing systems with finite buffers in the stationary state can be found e.g. in [4], [5] and [13]. In [7] distributions of three different characteristics for the system with the enqueueing process controlled by a drop function were investigated. A part of results from [7] was generalized in [14] and [15] where systems with bounded capacity and continuously distributed packet volumes were analyzed. Transient results for finite-buffer queues can also be found e.g. in monograph [1] and in papers [8] and [9]. In particular, in [9] the system with additional single server vacations is investigated.

Analytical results for distributions of the time to the buffer saturation, in fact, are mainly restricted to systems with a Poisson arrival process (simple or compound) or with the input flow described by a variant of MAP process. In [1] the case of the system with batch Poisson arrivals and constant service times was investigated. The compact formulae for the distribution of the time to the first buffer saturation was obtained in [2] for the $BMAP/G/1/N$ -type queue. The case

of the $MMPP$ -type input stream can be found in [3]. Some other results related with distributions of the buffer saturation period and the process of losses can be viewed e.g. in [6] and [12]. In particular, in [6] the representation for the joint transform of the busy period and numbers of packets being served and lost during the busy period was found for the system with phase-type distributions of interarrival and service times. In [12] the representation for the distribution of the number of buffer saturations during a single busy period was obtained for the system with Poisson arrivals.

In the paper distributions of times to successive buffer saturations are investigated in the $GI/M/1/N$ -type system with general independent input stream of packets and exponential service times. Applying the technique proposed in [10] and developed in [11] the compact formula for the Laplace transform of the distribution of the time to the first buffer saturation, conditioned by the number of packets present in the system at the opening, is obtained. For $k \geq 2$ a formula that allows to express the distribution of the time to the k th buffer saturation using the conditional distribution of the time to the first overflow is found.

So, the remaining part of the article is organized as follows. In the next Section 2 we present a mathematical model of the system and state auxiliary results. In Section 3 we present results for the distribution of the time to the first buffer saturation. In Section 4 a formula for computing distributions of times to next, successive buffer saturations via conditional distributions of the first one is obtained. Section 5 contains results for some special cases: the case of the system with Poisson arrival process and with one-place buffer. Section 6 is devoted to sample numerical computations, and the last Section 7 contains conclusions.

II. QUEUING MODEL AND AUXILIARY RESULTS

We consider the $GI/M/1/N$ -type queuing system in which interarrival times are independent and identically distributed random variables with a general-type distribution function $F(\cdot)$, and service times are exponentially distributed with mean μ^{-1} . The maximal system capacity equals N i.e. there are $N - 1$ places in the buffer queue and one place in service.

Let $F^{k*}(\cdot)$ be the k -fold Stieltjes convolution of the distribution function $F(\cdot)$ with itself. Besides, introduce the notation

$$f(s) = \int_0^{\infty} e^{-st} dF(t), \quad \operatorname{Re}(s) > 0. \quad (1)$$

Denote by $X(t)$ the number of packets present in the system at time t . Let $\beta_n^{(k)}$ be the time to the k th saturation of the buffer on condition that the system contains exactly n packets

at the opening. So, $\beta_n^{(k)}$ is the time between the completion epoch of the $(k-1)$ th period of buffer saturation and the initial moment of the k th period of saturation. Of course, $\beta_n^{(1)}$ denotes the time from the opening of the system to the first time of buffer overflow. Thus, we have

$$\beta_n^{(k)} = \inf \{t > 0 : X(t + \tau_{k-1}) = N \mid X(0) = n\}, \quad (2)$$

where $k \geq 1$, $0 \leq n \leq N-1$ and τ_k stands for the completion epoch of the k th period of buffer saturation (with additional agreement $\tau_0 = 0$).

For the investigation of the distribution of the time to the buffer saturation we propose the approach in which a certain specific-type system of equations occur. The solution of the system is found using the following result from [10] (see also [11]) :

Theorem 1. *Let (α_k) , $k \geq 0$, $\alpha_0 \neq 0$, and (φ_k) , $k \geq 1$, be known sequences.*

Every solution of the following system of equations:

$$\sum_{k=-1}^{n-1} \alpha_{k+1} x_{n-k} - x_n = \varphi_n, \quad n \geq 1, \quad (3)$$

can be written in the following form:

$$x_n = CR_n + \sum_{k=1}^n R_{n-k} \varphi_k, \quad n \geq 1, \quad (4)$$

where C does not depend on n and (R_k) is a sequence defined recursively by means of the given sequence (α_k) in the following way:

$$\begin{aligned} R_0 &= 0, \quad R_1 = \alpha_0^{-1}, \\ R_{k+1} &= R_1 \left(R_k - \sum_{i=0}^k \alpha_{i+1} R_{k-i} \right), \quad k \geq 1. \end{aligned} \quad (5)$$

The sequence (R_k) is called potential connected with the known sequence (α_k) .

In fact, in the paper we use a slightly modified version of Theorem 1, given below, where the system of equations is numbered beginning with $n = 2$. Indeed, the following corollary is a simple consequence of the last theorem:

Corollary 1. *Each solution of the following system of equations:*

$$\sum_{k=-1}^{n-2} \alpha_{k+1} x_{n-k} - x_n = \varphi_n, \quad n \geq 2, \quad (6)$$

can be written as

$$x_n = CR_{n-1} + \sum_{k=2}^n R_{n-k} \varphi_k, \quad n \geq 2, \quad (7)$$

where C is a constant independent on n and the potential (R_k) , $k \geq 0$, connected with the sequence (α_k) , is defined by the formulae (5).

III. DISTRIBUTION OF THE TIME TO THE FIRST BUFFER SATURATION

The main result of this section is a theorem that gives the explicit representation for the Laplace transform of the tail $B_n^{(1)}(\cdot)$ of conditional distribution function of the time $\beta_n^{(1)}$ from $t = 0$ to the moment of the first buffer saturation, for any value of initial system “contents” n .

Define

$$B_n^{(1)}(t) = \mathbf{P}\{\beta_n^{(1)} > t\}, \quad (8)$$

where $t > 0$, $0 \leq n \leq N-1$.

Since the arrival instants in the $GI/M/1$ -type queue are renewal (Markov) moments, then, applying the formula of total probability with respect to the first arrival moment after the opening of the system, we can write the following system of integral equations:

$$\begin{aligned} B_n^{(1)}(t) &= \sum_{k=0}^{n-1} \int_0^t \frac{(\mu x)^k}{k!} e^{-\mu x} B_{n-k+1}^{(1)}(t-x) dF(x) \\ &+ \sum_{k=n}^{\infty} \int_0^t \frac{(\mu x)^k}{k!} e^{-\mu x} B_1^{(1)}(t-x) dF(x) + (1-F(t)), \end{aligned} \quad (9)$$

where $0 \leq n \leq N-2$.

Let us comment briefly (9). The first summand on the right side of (9) describes the case in that the first packet arrives at time $x < t$ and before the first arrival the system does not empty completely. In consequence the number of packets at the first Markov moment x equals $n-k+1$, where k denotes the number of packets completely served before x , and the random event $\{\beta_n^{(1)} > t\}$ coincides with $\{\beta_{n-k+1}^{(1)} > t-x\}$. The second summand on the right side of (9) presents the situation in which the queue becomes empty before the first arrival epoch $x < t$. Thus, at the moment x the system “renews” its operation with exactly $n=1$ packet present. Obviously, if the first packet enters after t , then $\{\beta_n^{(1)} > t\}$ with probability one (compare the third summand on the right side of (9)).

Similarly, for $n = N-1$ we get

$$\begin{aligned} B_{N-1}^{(1)}(t) &= \sum_{k=1}^{N-2} \int_0^t \frac{(\mu x)^k}{k!} e^{-\mu x} B_{N-k}^{(1)}(t-x) dF(x) \\ &+ \sum_{k=N-1}^{\infty} \int_0^t \frac{(\mu x)^k}{k!} e^{-\mu x} B_1^{(1)}(t-x) dF(x) + (1-F(t)), \end{aligned} \quad (10)$$

Let us note that in the first summand on the right side of (10) the sum is taken from $k=1$. Evidently, in the situation of no completed services before the first arrival moment x , at time x the buffer becomes saturated. Since $x < t$, then the random event that the time to the first buffer saturation exceeds t has probability zero.

Let us introduce the following notation:

$$\alpha_k(s) = \int_0^{\infty} \frac{(\mu x)^k}{k!} e^{-(\mu+s)x} dF(x), \quad k \geq 0. \quad (11)$$

Moreover, let

$$B_n^{(1)}(s) = \int_0^{\infty} e^{-st} B_n^{(1)}(t) dt, \quad \text{Re}(s) > 0. \quad (12)$$

Now the system (9)–(10) can be transformed to the following form:

$$B_n^{(1)}(s) = \sum_{k=0}^{n-1} \alpha_k(s) B_{n-k+1}^{(1)}(s) + B_1^{(1)}(s) \sum_{k=n}^{\infty} \alpha_k(s) + \frac{1-f(s)}{s}, \quad 0 \leq n \leq N-2, \quad (13)$$

$$B_{N-1}^{(1)}(s) = \sum_{k=1}^{N-2} \alpha_k(s) B_{N-k}^{(1)}(s) + B_1^{(1)}(s) \sum_{k=N-1}^{\infty} \alpha_k(s) + \frac{1-f(s)}{s}. \quad (14)$$

Defining the following sequence:

$$\varphi_n(s) = -B_1^{(1)}(s) \sum_{k=n}^{\infty} \alpha_k(s) - \frac{1-f(s)}{s}, \quad n \geq 0, \quad (15)$$

the system (13) can be rewritten as

$$\sum_{k=-1}^{n-2} \alpha_{k+1}(s) B_{n-k}^{(1)}(s) - B_n^{(1)}(s) = \varphi_n(s), \quad (16)$$

where $0 \leq n \leq N-2$.

It is easy to note that the system (16) has the same form as (6). Of course, since now α_k and φ_n are, in general, functions of s , then the representation for the solution of (16) should be written as (see (7)):

$$B_n^{(1)}(s) = C(s) R_{n-1}(s) + \sum_{k=2}^n R_{n-k}(s) \varphi_k(s), \quad (17)$$

where $n \geq 2$, the function $C(s)$ is independent on n and the potential $R_k(s)$ is defined in (5), where now $\alpha_k = \alpha_k(s)$.

Of course, at this stage it is impossible to write down the formulae for $B_n^{(1)}$, $0 \leq n \leq N-1$, explicitly since the expressions for $C(s)$, $B_0(s)$ and $B_1(s)$ are unknown (the formula (17) is valid for $n \geq 2$). To find them substitute firstly $n = 2$ into the equation (17). We obtain

$$B_2^{(1)}(s) = C(s) R_1(s) = \frac{C(s)}{\alpha_0(s)}. \quad (18)$$

Taking $n = 1$ in (13) and applying the identities

$$\alpha_0(s) = f(s + \mu) \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha_k(s) = f(s), \quad (19)$$

we get

$$B_1^{(1)}(s) = \alpha_0(s) B_2^{(1)}(s) + B_1^{(1)}(s) (f(s) - f(s + \mu)) + \frac{1-f(s)}{s}, \quad (20)$$

Now, substituting (18) into (20), we have

$$B_1^{(1)}(s) = \frac{s^{-1}(1-f(s)) + C(s)}{1-f(s) + f(s + \mu)}. \quad (21)$$

Similarly, taking $n = 0$ in (13), we obtain

$$B_0^{(1)}(s) = B_1^{(1)}(s) f(s) + \frac{1-f(s)}{s}. \quad (22)$$

Let us now substitute $n = N-1$ into the formula (17). Applying, additionally, the representations (15) and (21) we get

$$B_{N-1}^{(1)}(s) = C(s) R_{N-2}(s) - \sum_{k=2}^{N-1} R_{N-1-k}(s) \times \left(\frac{s^{-1}(1-f(s)) + C(s)}{1-f(s) + f(s + \mu)} \sum_{i=k}^{\infty} \alpha_i(s) + \frac{1-f(s)}{s} \right). \quad (23)$$

Another representation for $B_{N-1}^{(1)}(s)$ can be found using the formula (14). Substituting the identity (17) into (14) we have

$$B_{N-1}^{(1)}(s) = \sum_{k=1}^{N-2} \alpha_k(s) \left[C(s) R_{N-1-k}(s) - \sum_{i=2}^{N-k} R_{N-k-i}(s) \times \left(\frac{s^{-1}(1-f(s)) + C(s)}{1-f(s) + f(s + \mu)} \sum_{j=i}^{\infty} \alpha_j(s) + \frac{1-f(s)}{s} \right) \right] + \frac{s^{-1}(1-f(s)) + C(s)}{1-f(s) + f(s + \mu)} \sum_{k=N-1}^{\infty} \alpha_k(s) + \frac{1-f(s)}{s}. \quad (24)$$

Now, we can find easily the formula for $C(s)$, comparing the right sides of (23) and (24). Indeed, we obtain

$$C(s) = \frac{s^{-1}(1-f(s))(1+a(s)-d(s)+g(s))}{h(s)+d(s)-a(s)}, \quad (25)$$

where

$$a(s) = \frac{\sum_{k=2}^{N-1} R_{N-1-k}(s) \sum_{i=k}^{\infty} \alpha_i(s) + \sum_{k=N-1}^{\infty} \alpha_k(s)}{1-f(s) + f(s + \mu)}, \quad (26)$$

$$d(s) = \frac{\sum_{k=1}^{N-2} \alpha_k(s) \sum_{i=2}^{N-k} R_{N-k-i}(s) \sum_{j=i}^{\infty} \alpha_j(s)}{1-f(s) + f(s + \mu)}, \quad (27)$$

$$g(s) = \sum_{k=2}^{N-1} R_{N-1-k}(s) - \sum_{k=1}^{N-2} \alpha_k(s) \sum_{i=2}^{N-k} R_{N-k-i}(s) \quad (28)$$

and

$$h(s) = R_{N-2}(s) - \sum_{k=1}^{N-2} \alpha_k(s) R_{N-k-1}(s). \quad (29)$$

Putting together the formulae (17), (21), (22) and (25) we obtain the following main theorem:

Theorem 2. *The formula for the Laplace transform of the tail of conditional distribution of the time $\beta_n^{(1)}$ to the first buffer saturation in the GI/M/1/N-type queue is following:*

$$B_n^{(1)}(s) = \int_0^{\infty} e^{-st} \mathbf{P}\{\beta_n^{(1)} > t\} dt = \frac{s^{-1}(1-f(s))(1+a(s)-d(s)+g(s))}{h(s)+d(s)-a(s)} R_{n-1}(s) + \sum_{k=2}^n R_{n-k}(s) \varphi_k(s), \quad (30)$$

where $2 \leq n \leq N - 1$, and

$$\begin{aligned} \varphi_n(t) = & \\ & - \frac{s^{-1}(1-f(s))(1+g(s)+h(s))}{(1-f(s)+f(s+\mu))(h(s)+d(s)-a(s))} \sum_{k=n}^{\infty} \alpha_k(s) \\ & - \frac{1-f(s)}{s}. \end{aligned} \quad (31)$$

Besides

$$B_1^{(1)}(s) = \frac{s^{-1}(1-f(s))(1+g(s)+h(s))}{(1-f(s)+f(s+\mu))(h(s)+d(s)-a(s))} \quad (32)$$

and

$$B_0^{(1)}(s) = B_1^{(1)}(s)f(s) + \frac{1-f(s)}{s}. \quad (33)$$

The representations for $R_k(s)$, $\alpha_k(s)$, $a(s)$, $d(s)$, $g(s)$ and $h(s)$ are given in (5), (11), (26), (27), (28) and (29) respectively.

IV. DISTRIBUTION OF THE TIME TO THE k TH BUFFER SATURATION FOR $k \geq 2$

The main aim of this section is in finding a formula to express the distribution (tail) of the time to the k th buffer saturation, for $k \geq 2$, defined as

$$B^{(k)}(t) = \mathbf{P}\{\beta^{(k)} > t\}, \quad (34)$$

in terms of conditional distributions (tails) $B_n^{(1)}(\cdot)$ of the time to the first saturation. Indeed, below we prove the following theorem:

Theorem 3. *In the GI/M/1/N-type system the tail $B^{(k)}(\cdot)$ of the distribution function of the time to the k th buffer saturation, for $k \geq 2$, is independent on k and on the initial state of the system, and can be written in terms of conditional distributions $B_n^{(1)}(\cdot)$ of the time to the first saturation in the following way:*

$$\begin{aligned} B^{(k)}(t) = & \int_0^{\infty} dx \sum_{j=0}^{\infty} \int_0^x e^{-\mu y} dF^{j*}(y) \\ & \times \int_{x-y}^{x-y+t} \left[\sum_{i=1}^{N-2} \frac{\mu^{i+1}(u-x+y)^i}{i!} e^{-\mu u} \right. \\ & \times B_{N-i}^{(1)}(t-u+x-y) \\ & + \sum_{i=N-1}^{\infty} \frac{\mu^{i+1}(u-x+y)^i}{i!} e^{-\mu u} \\ & \left. \times B_1^{(1)}(t-u+x-y) \right] dF(u) \\ & + \mu \int_0^{\infty} e^{-\mu x} dx \sum_{j=0}^{\infty} \int_0^x (1-F(x-y+t)) dF^{j*}(y). \end{aligned} \quad (35)$$

Proof:

The independence of $B^{(k)}$ on k and on the number of packets present in the system initially is evident. Indeed, $\beta^{(k)}$ expresses the time from the completion epoch of the

$(k-1)$ th buffer overflow period to the initial epoch of the k th one. But, independently on the initial state of the system, at the completion epoch of each period of buffer saturation the number of packets equals $N-1$ (due to ‘‘individual’’ service discipline).

To prove the representation (35) let us apply the formula of total probability with respect to the first arrival epoch after the buffer saturation period. Such an approach is fully justified. Indeed, in the GI/M/1/N-type queuing model, due to general distributions of interarrival times, the completion instants τ_1, τ_2, \dots of successive periods of buffer saturation are not Markov moments. Simultaneously, due to the memoryless property of exponential distribution, each period of buffer saturation is exponentially distributed with the same mean as the service time i.e. μ^{-1} .

So, the following identity holds true:

$$\begin{aligned} B_n^{(2)}(t) = & \int_0^{\infty} \mu e^{-\mu x} dx \sum_{j=0}^{\infty} \int_0^x dF^{j*}(y) \\ & \times \int_{x-y}^{x-y+t} \left[\sum_{i=1}^{N-2} \frac{[\mu(u+y-x)]^i}{i!} e^{-\mu(u+y-x)} \right. \\ & \times B_{N-i}^{(1)}(t-(u+y-x)) \\ & + \sum_{i=N-1}^{\infty} \frac{[\mu(u+y-x)]^i}{i!} e^{-\mu(u+y-x)} \\ & \left. \times B_1^{(1)}(t-(u+y-x)) \right] dF(u) \\ & + \int_0^{\infty} \mu e^{-\mu x} dx \sum_{j=0}^{\infty} \int_0^x (1-F(x-y+t)) dF^{j*}(y). \end{aligned} \quad (36)$$

Let us explain the last formula in details. We position the origin of the time axis at the initial moment of the buffer saturation period (that is a Markov moment since it is connected with an arrival of a packet). On the right side of (36) x indicates the completion epoch of this period and y is the moment of the last arrival before x . In the first summand on the right side of (36) the next packet (arriving after y) occurs after time $u < x-y+t$ and hence the system does not empty between x and $u-x+y$. The second summand relates to the situation in which the server becomes idle before time period u . The last, third summand on the right side of (36) describes the situation in which we have no arrivals during time t , beginning with the completion epoch of the buffer saturation period. In consequence, independently on the instantaneous number of packets in the system, the next period of buffer saturation will start after time t with probability one.

Simplification of (36) immediately leads to (35) \square

V. SPECIAL CASES

In this section we present some results for two special cases of the considered queuing model: the case of simple Poisson arrival process and the case of the buffer with one place only.

A. System with Poisson arrivals

Let us consider the M/M/1/N-type queuing model in which the arriving packets occur according to a Poisson

process with intensity λ . As it turns out, in such a case it is possible to express the potential (R_k) , defined in (5) and connected with the given sequence $(\alpha_k(s))$ from (11), in the explicit form.

Firstly, let us note that now we have

$$\alpha_k(s) = \lambda \int_0^\infty \frac{(\mu x)^k}{k!} e^{-(\lambda+\mu+s)x} dx = \frac{\lambda \mu^k}{(\lambda + \mu + s)^{k+1}} \quad (37)$$

and, of course, $f(s) = \frac{\lambda}{\lambda+s}$.

Define the generating function $R(s, z)$ of the potential $(R_k(s))$ in the following way:

$$R(s, z) = \sum_{k=0}^{\infty} z^k R_k(s), \quad |z| < 1. \quad (38)$$

Taking into consideration the definition (5) we obtain

$$\begin{aligned} R(s, z) &= \sum_{k=1}^{\infty} z^k R_k(s) = zR_1(s) \\ &+ \sum_{k=1}^{\infty} z^{k+1} \left[R_1(s)(R_k(s) - \sum_{i=0}^k \alpha_{i+1}(s)R_{k-i}(s)) \right] \\ &= zR_1(s)(1 + R(s, z)) \\ &- R_1(s) \sum_{i=0}^{\infty} \alpha_{i+1}(s) z^{i+1} \sum_{k=i}^{\infty} z^{k-i} R_{k-i}(s) \\ &= zR_1(s)(1 + R(s, z)) \\ &- R_1(s)R(s, z)[f(s + \mu(1-z)) - \alpha_0(s)]. \end{aligned} \quad (39)$$

Now we eliminate $R(s, z)$ as follows:

$$R(s, z) = \frac{z}{f(s + \mu(1-z)) - z} \quad (40)$$

and, in the case of the considered $M/M/1/N$ -type system, we get

$$R(s, z) = \frac{z[\lambda + s + \mu(1-z)]}{\lambda - z[\lambda + s + \mu(1-z)]}. \quad (41)$$

To find $R_k(s)$ in the explicit form we can use the *Mathematica* environment, e.g. the function *InverseZTransform*, substituting firstly on the right side of (41) $z = \frac{1}{z}$.

Indeed, the k th term of the potential can be written in the following form:

$$\begin{aligned} R_k(s) &= \frac{1}{2\sigma(s)(2\lambda)^k} \left\{ 2\kappa(s) \left[(\kappa(s) + \sigma(s))^n \right. \right. \\ &- (\kappa(s) - \sigma(s))^n \left. \right] \\ &+ I\{k \geq 1\} \left[(\kappa(s) + \sigma(s))(\kappa(s) - \sigma(s))^n \right. \\ &- (\kappa(s) - \sigma(s))(\kappa(s) + \sigma(s))^n \left. \right] \left. \right\}, \end{aligned} \quad (42)$$

where

$$\kappa(s) = s + \lambda + \mu \quad (43)$$

and

$$\sigma(s) = \sqrt{s^2 + (\lambda - \mu)^2 + 2s(\lambda + \mu)}. \quad (44)$$

B. System with one-place buffer

Let us consider the original $GI/M/1/N$ -type system with $N = 2$ i.e. with the one-place buffer only.

It is easy to verify that the system of equations (13)–(14) can be written now in the following form:

$$\begin{cases} B_0^{(1)}(s) = f(s)B_1^{(1)}(s) + \frac{1-f(s)}{s}, \\ B_1^{(1)}(s) = B_1^{(1)}(s)(f(s) - f(s + \mu)) + \frac{1-f(s)}{s}. \end{cases} \quad (45)$$

Hence we eliminate $B_0^{(1)}(s)$ and $B_1^{(1)}(s)$ as follows:

$$\begin{cases} B_0^{(1)}(s) = \frac{(1-f(s))(1+f(s+\mu))}{s(1-f(s)+f(s+\mu))}, \\ B_1^{(1)}(s) = \frac{1-f(s)}{s(1-f(s)+f(s+\mu))}. \end{cases} \quad (46)$$

From the definitions (8) and (12) follows that

$$\mathbf{E}\beta_n^{(1)} = B_n^{(1)}(0). \quad (47)$$

Besides, let us note that

$$\lim_{s \rightarrow 0} \frac{1-f(s)}{s} = -\lim_{s \rightarrow 0} f'(s) = \mathbf{E}F, \quad (48)$$

where $\mathbf{E}F$ denotes the mean of interarrival times.

Now the identities (45) easily lead to

$$\mathbf{E}\beta_0^{(1)} = \mathbf{E}F f^{-1}(\mu)(1 + f(\mu)) \quad (49)$$

and

$$\mathbf{E}\beta_1^{(1)} = \mathbf{E}F f^{-1}(\mu), \quad (50)$$

thus

$$\mathbf{E}\beta_0^{(1)} = \mathbf{E}F + \mathbf{E}\beta_0^{(1)}. \quad (51)$$

The formula (51) has, obviously, very simple intuitive explanation: in the system with one-place buffer the mean of the time to the first buffer saturation in the system that begins its evolution being empty ($\mathbf{E}\beta_0^{(1)}$), is a sum of the mean of a waiting time for the first arriving packet ($\mathbf{E}F$) and the mean of the time to the first buffer saturation with one packet present ($\mathbf{E}\beta_1^{(1)}$). Of course, similar conclusions can be obtained in general case, basing on equations of the system (13)–(14).

In the case of the Poisson process with intensity λ describing the input flow of packets the formulae (49)–(50) simplify to the forms

$$\mathbf{E}\beta_0^{(1)} = \frac{2\lambda + \mu}{\lambda^2} \quad \text{and} \quad \mathbf{E}\beta_1^{(1)} = \frac{\lambda + \mu}{\lambda^2}. \quad (52)$$

VI. NUMERICAL RESULTS

In this section we present sample numerical computations illustrating theoretical results obtained in Theorem 2. All computations are executed using the *Mathematica* environment.

Example 1.

Let us take into consideration the $E_3/M/1/3$ -type system in which interarrival times have 3-Erlang distributions with parameter λ i.e.

$$F(t) = 1 - e^{-\lambda t} \left(1 + \lambda t + \frac{1}{2} \lambda^2 t^2 \right), \quad t > 0. \quad (53)$$

The traffic load of the system equals $\rho = \frac{\lambda}{3\mu}$, where μ denotes the service rate.

In Table I the values of $\mathbf{E}\beta_0^{(1)}$, $\mathbf{E}\beta_1^{(1)}$ and $\mathbf{E}\beta_2^{(1)}$ are presented for different values of parameter λ and, in consequence, for different values of the traffic load ρ , decreasing from 4.00 to 0.17. Results in Table I are obtained keeping $\mu = 1$.

TABLE I
MEAN TIME TO BUFFER SATURATION IN $E_3/M/1/3$ QUEUE IN FUNCTION OF ARRIVAL RATE

No.	Parameter λ	Traffic load ρ	$\mathbf{E}\beta_0^{(1)}$	$\mathbf{E}\beta_1^{(1)}$	$\mathbf{E}\beta_2^{(1)}$
1	12.0	4.00	0.89862	0.64862	0.33077
2	9.0	3.00	1.28063	0.94730	0.49005
3	7.0	2.33	1.78334	1.35477	0.71504
4	5.0	1.67	2.90999	2.30999	1.27319
5	4.0	1.33	4.19696	3.44696	1.98212
6	3.0	1.00	7.21125	6.21125	3.84088
7	2.5	0.83	10.7058	9.50584	6.21304
8	2.0	0.67	18.5859	17.0859	12.0234
9	1.0	0.33	183	180	156
10	0.5	0.17	4218	4212	4050

Obviously, as one can note, the values of the mean of the time to the first buffer saturation increase with decreasing traffic load ρ . As it is intuitively clear, the values of $\mathbf{E}\beta_2^{(1)}$ are essentially smaller than the corresponding values of $\mathbf{E}\beta_1^{(1)}$ and $\mathbf{E}\beta_0^{(1)}$: in the case of $n = 2$ packets present in the system initially, the buffer saturation begins if only the first arrival instant "outruns" the first service completion epoch.

Results from Table I are presented geometrically in Figure 1 (nos. 1–8), where dotted, dashed and solid lines correspond to the values of $\mathbf{E}\beta_0^{(1)}$, $\mathbf{E}\beta_1^{(1)}$ and $\mathbf{E}\beta_2^{(1)}$ respectively.

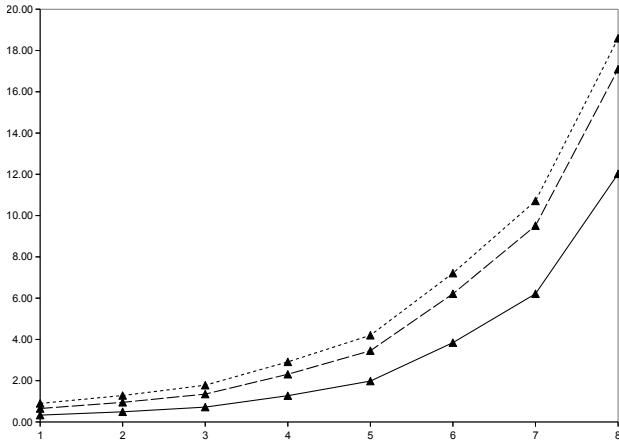


Fig. 1. Mean time to buffer saturation in $E_3/M/1/3$ queue in function of arrival rate (nos. 1-8)

In Table II (see also Figure 2) the values of the mean time to the first buffer saturation, under different initial conditions of the system, are presented in function of the service rate μ . Results in Table II are obtained taking $\lambda = 3$ (so $\rho = \frac{1}{\mu}$).

Example 2.

Let us consider now the $D/M/1/3$ -type queuing system in which interarrival times have deterministic distributions i.e. the arriving packets occur at constant intervals equal Δ . Assume

TABLE II
MEAN TIME TO BUFFER SATURATION IN $E_3/M/1/3$ QUEUE IN FUNCTION OF SERVICE RATE

No.	Service rate μ	Traffic load ρ	$\mathbf{E}\beta_0^{(1)}$	$\mathbf{E}\beta_1^{(1)}$	$\mathbf{E}\beta_2^{(1)}$
1	0.1	10.00	3.21402	2.21402	1.11065
2	0.3	3.33	3.73956	2.73956	1.40856
3	0.5	2.00	4.42903	3.42903	1.84107
4	0.6	1.67	4.84998	3.84998	2.12198
5	0.7	1.43	5.33077	4.33077	2.45474
6	0.8	1.25	5.87897	4.87897	2.84667
7	0.9	1.11	6.50281	5.50281	3.30581
8	1.0	1.00	7.21125	6.21125	3.84088
9	1.5	0.67	12.3906	11.3906	8.01563
10	2.0	0.50	21.5075	20.5075	15.8779

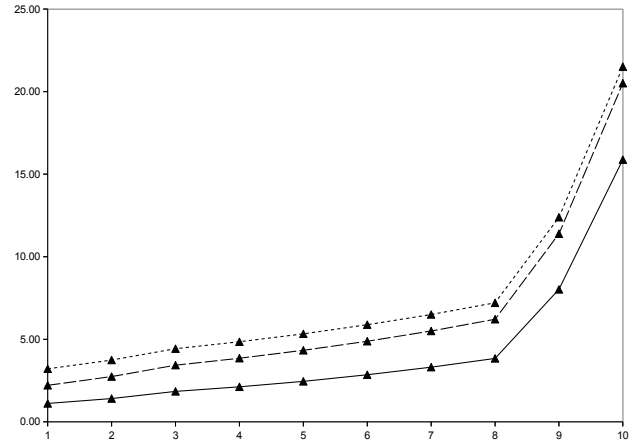


Fig. 2. Mean time to buffer saturation in $E_3/M/1/3$ queue in function of service rate

that the service rate equals $\mu = 2$, so the traffic load in the system can be expressed as $\rho = \frac{1}{2\Delta}$.

In Table III the values of $\mathbf{E}\beta_0^{(1)}$, $\mathbf{E}\beta_1^{(1)}$ and $\mathbf{E}\beta_2^{(1)}$ are presented for 10 different values of the constant interarrival times Δ , namely from 0.1 to 1.0 with step 0.1. Let us note that, simultaneously, the traffic load ρ changes from 5 to 0.5, respectively, thus the regime of the system operation changes from the overloaded to the underloaded one.

TABLE III
MEAN TIME TO BUFFER SATURATION IN $D/M/1/3$ QUEUE IN FUNCTION OF ARRIVAL RATE

No.	Parameter Δ	Traffic load ρ	$\mathbf{E}\beta_0^{(1)}$	$\mathbf{E}\beta_1^{(1)}$	$\mathbf{E}\beta_2^{(1)}$
1	0.1	5.00	0.34690	0.24690	0.12475
2	0.2	2.50	0.82413	0.62413	0.32576
3	0.3	1.67	1.51469	1.21469	0.66805
4	0.4	1.25	2.55926	2.15926	1.26904
5	0.5	1.00	4.19453	3.69453	2.33539
6	0.6	0.83	6.81549	6.21549	4.22342
7	0.7	0.71	11.0758	10.3758	7.53716
8	0.8	0.63	18.0486	17.2486	13.2861
9	0.9	0.56	29.4827	28.5827	23.1380
10	1.0	0.50	48.2091	47.2091	39.8200

The interpretation of results in Table III is similar to that in Table I. Additionally, let us observe in practice the realization of the equation (51): in the case of the $D/M/1/N$ -type system we have $\mathbf{E}\beta_1^{(1)} = \mathbf{E}\beta_0^{(1)} + \Delta$ (see remark on the equation (51)). Results from Table III are presented geometrically

in Figure 3, where, as previously, dotted, dashed and solid lines correspond to the values of $E\beta_0^{(1)}$, $E\beta_1^{(1)}$ and $E\beta_2^{(2)}$ respectively.

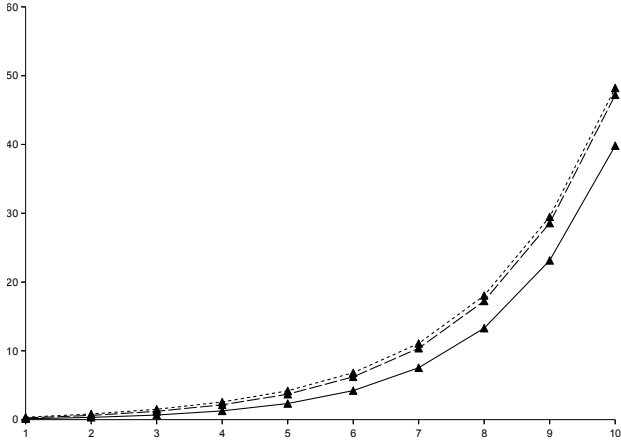


Fig. 3. Mean time to buffer saturation in $D/M/1/3$ queue in function of arrival rate

Finally, in Table IV (see also Figure 4) the values of the mean time to the first buffer saturation, under three different initial conditions of the operation of the system, are given in function of the service rate μ . Results in Table IV are obtained taking $\Delta = 2$ (so $\rho = \frac{1}{2\mu}$).

TABLE IV
MEAN TIME TO BUFFER SATURATION IN $D/M/1/3$ QUEUE IN FUNCTION OF SERVICE RATE

No.	Service rate μ	Traffic load ρ	$E\beta_0^{(1)}$	$E\beta_1^{(1)}$	$E\beta_2^{(1)}$
1	0.1	5.00	6.93789	4.93789	2.49509
2	0.2	2.50	8.24127	6.24127	3.25762
3	0.3	1.67	10.0979	8.09793	4.45369
4	0.4	1.25	12.7963	10.7963	6.34520
5	0.5	1.00	16.7781	14.7781	9.34155
6	0.6	0.83	22.7183	20.7183	14.0781
7	0.7	0.71	31.6451	29.6451	21.5347
8	0.8	0.63	45.1214	43.1214	33.2154
9	0.9	0.56	65.5170	63.5170	51.4177
10	1.0	0.50	96.4182	94.4182	79.6401

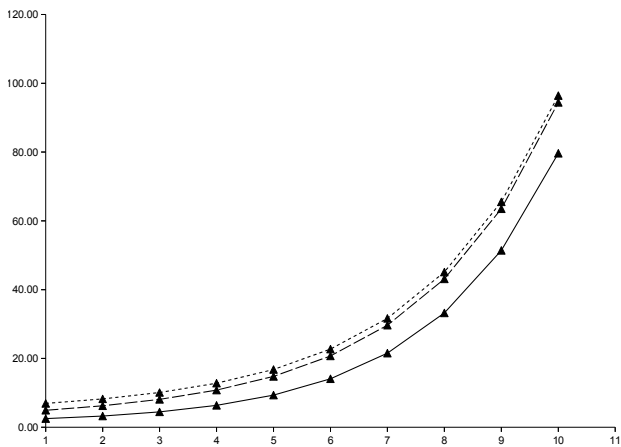


Fig. 4. Mean time to buffer saturation in $D/M/1/3$ queue in function of service rate

VII. CONCLUSION

In the article distributions of times to successive buffer saturations are analyzed in the $GI/M/1/N$ -type finite-buffer queue with general independent input stream of packets and exponentially distributed service times. The method of potential is applied to obtain the representation for the Laplace transform of the conditional distribution of the time to the first period of buffer saturation. Moreover, a formula for the tail of the distribution function of the “waiting” time to reaching the k th saturation of the buffer, for $k \geq 2$, is proved. The formula expresses the distribution of the time to next buffer overflows in terms of the conditional distribution of the time to the first buffer saturation. Some important special systems are discussed separately, namely the case of the system with Poisson arrival stream of packets and with one-place buffer. Numerical examples illustrating theoretical results are included as well, where the cases of the 3-Erlang and deterministic distributions of interarrival times are investigated.

REFERENCES

- [1] A. Chydzinski, R. Winiarczyk, “Distribution of the first buffer overflow time in a deterministic service time queue,” in *Computers and Communications, 2005. ISCC 2005. Proceedings of IEEE symposium on Computers and Communications*, 2005, pp. 637–641.
- [2] A. Chydzinski, “Time to reach buffer capacity in a $BMAP$ queue,” *Stochastic Models*, vol. 23, pp. 195–209, 2007.
- [3] A. Chydzinski, “Time to buffer overflow in an $MMPP$ queue,” *Lecture Notes in Computer Science*, vol. 4479, pp. 879–889, 2007.
- [4] J. W. Cohen, *The Single Server Queue*. Amsterdam - New York - Oxford: North-Holland Publishing Company, 1982.
- [5] R. B. Cooper, *Introduction To Queueing Theory*, North Holland, 1981.
- [6] A. Al Hanbali, “Busy period analysis of the level dependent $PH/PH/1/K$ queue,” *Queueing Syst.*, vol. 67, pp. 221–249, 2011.
- [7] W. M. Kempa, “On main characteristics of the $M/M/1/N$ queue with single and batch arrivals and the queue size controlled by AQM algorithms,” *Kybernetika*, vol. 47, no. 6, pp. 930–943, 2011.
- [8] W. M. Kempa, “Departure process in finite-buffer queue with batch arrivals,” *Lecture Notes Compt. Science*, vol. 6751, pp. 1–13, 2011.
- [9] W. M. Kempa, “The virtual waiting time in a finite-buffer queue with a single vacation policy,” *Lecture Notes Compt. Science*, vol. 7314, pp. 47–60, 2012.
- [10] V. S. Korolyuk, *Boundary-value problems for complicated Poisson processes*. Kiev: Naukova Dumka, 1975 (in Russian).
- [11] V. S. Korolyuk, M. S. Bratiichuk, B. Pirdzhanov, *Boundary-value problems for random walks*. Ashkhabad: Ylym, 1987 (in Russian)
- [12] E. Y. Lee, K. K. J. Kinatader, “The expected wet period of finite dam with exponential inputs,” *Stochastic Processes and Applications*, vol. 90, pp. 175–180, 2000.
- [13] H. Takagi, *Queueing Analysis, vol. 2. Finite Systems*. Amsterdam: North-Holland, 1993.
- [14] O. Tikhonenko, W. M. Kempa, “The generalization of AQM algorithms for queueing systems with bounded capacity,” *Lecture Notes in Computer Science*, vol. 7204, 242–251, 2012.
- [15] O. Tikhonenko, W. M. Kempa, “Queue-size distribution in $M/G/1$ -type system with the bounded capacity and packet dropping,” *Lect. Notes Commun. Compt. Inf. Science* (accepted for publication)



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