Vector Stochastic Differential Equations Used to Electrical Networks with Random Parameters

Edita Kolářová and Lubomír Brančík

***Abstract*—In this paper we present an application of the Itô stochastic calculus to the problem of modelling RLC electrical circuits. The deterministic model of the circuit is replaced by a stochastic model by adding a noise term to various parameters of the circuit. The analytic solutions of the resulting stochastic integral equations are found using the multidimensional Itô formula. For the numerical simulations in the examples we used MATLAB®. The SDE approach has its perspectives in the simulation even higher-order circuits representing more complex physical systems, as their real implementations are often subject to a number of random effects. An example for a transmission line lumped-parameter model is provided.**

***Keywords*—Stochastic differential equations, Wiener process, Itô formula, electrical network, transmission line model.**

I. Introduction

Stochastic differential equations (SDEs) describe systems including some random effects. In this paper we deal with vector Itô stochastic differential equations and refer to some utilization in electrical engineering simulations. Using the Itô formula we find the analytic solution of the equations containing one stochastic parameter. Then we apply the theory to the stochastic model of an RLC electrical network that we get by replacing one of its parameters in the deterministic model by a random process. The theory of SDEs can find its place in various fields of the science and engineering, when random effects are to be considered [1]-[4]. In field of the electrical engineering it can cover a number of random processes occurring in electrical systems, see e.g. [5]-[7] to mention at least a few application areas. Some attention is still paid to the first-order RL or RC circuits, see e.g. [8]-[11], where suitable numerical techniques and different noise types have been studied. From practical point of view, however, the second-order RLC or RLGC circuits are of major importance as they serve as building blocks of more complex physical models. In this paper we want to prepare a bases for the solution of interconnects under stochastically varied parameters, modelled by a cascade connection of just the 2nd-order networks [12], [13] and do first verifications. Whilst for the 1st-order models a scalar SDEs theory is commonly used, see [9], in case of higher-order models the vector SDEs are needed to be applied.

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Edita Kolářová is with Brno University of Technology, Department of Mathematics, Technicka 8, Brno, Czech Republic. (e-mail: kolara@feec.vutbr.cz).

Lubomír Brančík is with Brno University of Technology, Department of Radio Electronics, Purkynova 118, Brno, Czech Republic. (e-mail: brancik@feec.vutbr.cz).

II. Stochastic Differential Equations

A. Vector Stochastic Differential Equations

In the theory of stochastic differential equations the Wiener process plays a very important role, because it represents the integral of the so called Gaussian white noise, that describes the randomness in stochastic models of physical events.

A real-valued Wiener process $W(t)$ is a continuous stochastic process with independent increments, $W(0) = 0$ and $W(t) - W(s)$ distributed $N(0, t-s),\ 0 \le s < t$. Notice in particular that $E[W(t)] = 0,\ E[W^2(t)] = t,\ t \ge 0$.

We can define an N-dimensional SDE in vector form as

$$\mathrm{d}\mathbf{X}(t) = \widetilde{\mathbf{A}}(t, \mathbf{X}(t))\,\mathrm{d}t + \sum_{j=1}^{M} \widetilde{\mathbf{B}^j}(t, \mathbf{X}(t))\,\mathrm{d}W^j(t), \quad (1)$$

where $\widetilde{\mathbf{A}} : \langle 0, T\rangle \times \mathbb{R}^N \to \mathbb{R}^N$ is a vector function, $\widetilde{\mathbf{B}^j}$ represents the $j-$th column of the matrix function $\widetilde{\mathbf{B}} : \langle 0, T\rangle \times \mathbb{R}^N \to \mathbb{R}^{N\times M}$ and $\mathrm{d}\mathbf{W}(t) = (\mathrm{d}W^1(t), \ldots, \mathrm{d}W^M(t))$ is a column vector, where $W^1(t), \ldots, W^M(t)$ are independent Wiener processes representing the noise. The solution is a stochastic vector process $\mathbf{X}(t) = (X^1(t), \ldots, X^N(t))$. By an SDE we understand in fact an integral equation

$$\mathbf{X}(t) = \mathbf{X}_0 + \int_{t_0}^{t} \widetilde{\mathbf{A}}(s, \mathbf{X}(s))\,\mathrm{d}s + \sum_{j=1}^{M} \int_{t_0}^{t} \widetilde{\mathbf{B}^j}(s, \mathbf{X}(s))\,\mathrm{d}W^j(s), \quad (2)$$

where the integral with respect to $\mathrm{d}s$ is the Lebesgue integral and the integrals with respect to $\mathrm{d}W^j(s)$ are stochastic integrals, called the Itô integrals (see [2]).

Although the Itô integral has some very convenient properties, the usual chain rule of classical calculus doesn't hold. The appropriate stochastic chain rule is known as the Itô formula.

B. The Multidimensional Itô Formula

Let the stochastic process $\mathbf{X}(t)$ be a solution of the vector stochastic differential equation (1) for some suitable functions $\mathbf{A}, \mathbf{B}$ (see [2], p.48). Let $\mathbf{g}(t, \mathbf{X}) : (0, \infty) \times \mathbf{R}^N \to \mathbf{R}^P$ be a twice continuously differentiable function. Then

$$\mathbf{Y}(t) = \mathbf{g}(t, \mathbf{X}(t)) = (g_1(t, \mathbf{X}), \ldots, g_P(t, \mathbf{X})) \quad (3)$$

is a stochastic process, whose $k-$th component is given by

$$dY^k = \frac{\partial g_k}{\partial t}(t, \mathbf{X})\,\mathrm{d}t + \sum_i \frac{\partial g_k}{\partial x_i}(t, \mathbf{X})\,\mathrm{d}X^i + \quad (4)$$

$$+\frac{1}{2}\sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, \mathbf{X})(\,\mathrm{d}X^i)(\,\mathrm{d}X^j),$$

where $\mathrm{d}X^i\cdot\,\mathrm{d}X^j$ is computed according to the rules $\mathrm{d}t\cdot\,\mathrm{d}t = \mathrm{d}t\cdot\,\mathrm{d}W^i = \mathrm{d}W^i\cdot\,\mathrm{d}t = 0$ and $\mathrm{d}W^i\cdot\,\mathrm{d}W^j = \delta_{i,j}\,\mathrm{d}t$.

C. Vector Linear Itô Stochastic Differential Equations

The stochastic differential equation (1) is called linear, provided the coefficients have the form

$$\widetilde{\mathbf{A}}(t,\mathbf{X}(t)) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{a}(t), \tag{5}$$

and

$$\widetilde{\mathbf{B}^j}(t,\mathbf{X}(t)) = \mathbf{B}^j(t)\,\mathbf{X}(t) + \mathbf{b}^j(t), \quad j = 1\ldots M \tag{6}$$

for $\mathbf{A}$, $\mathbf{B}^j : \langle 0,T\rangle \to \mathbb{R}^{N\times N}$ and $\mathbf{a}$, $\mathbf{b}^j : \langle 0,T\rangle \to \mathbb{R}^N$.

For the applications to the RLC(G) circuit we need only a 2 dimensional vector linear equation with a one dimensional Wiener process, which has the form

$$\mathrm{d}\mathbf{X}(t) = \big(\mathbf{A}\,\mathbf{X}(t) + \mathbf{a}(t)\big)\,\mathrm{d}t + \big(\mathbf{B}\,\mathbf{X}(t) + \mathbf{b}(t)\big)\,\mathrm{d}W(t), \tag{7}$$

where $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices, $\mathbf{a} : \langle 0,T\rangle \to \mathbb{R}^2$ and similarly $\mathbf{b} : \langle 0,T\rangle \to \mathbb{R}^2$ are vector functions, $W(t)$ is the Wiener process.

D. Solution with Additive Noise

First we solve the equation (7) with additive noise, which is the case when $\mathbf{B} \equiv 0$. To find the analytic solution of the equation

$$\mathrm{d}\mathbf{X}(t) = \big(\mathbf{A}\,\mathbf{X}(t) + \mathbf{a}(t)\big)\,\mathrm{d}t + \mathbf{b}(t)\,\mathrm{d}W(t), \tag{8}$$

we use the multidimensional Itô formula and find the derivative of the function $g(t,\mathbf{X}(t)) = \mathrm{e}^{-\mathbf{A}t}\mathbf{X}(t) : \langle 0,T\rangle \times \mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{aligned}
\mathrm{d}g(t,\mathbf{X}(t)) = {}& \mathrm{d}\left(\mathrm{e}^{-\mathbf{A}\,t}\cdot \mathbf{X}(t)\right) = \\
= -\mathbf{A}\mathrm{e}^{-\mathbf{A}\,t}\cdot \mathbf{X}(t)\,\mathrm{d}t &+ \mathrm{e}^{-\mathbf{A}\,t}\cdot\,\mathrm{d}\mathbf{X}(t) - 0\cdot(\,\mathrm{d}\mathbf{X}(t))^2 = \\
= -\mathbf{A}\mathrm{e}^{-\mathbf{A}\,t}\mathbf{X}(t)\,\mathrm{d}t &+ \mathrm{e}^{-\mathbf{A}\,t}\mathbf{A}\mathbf{X}(t)\,\mathrm{d}t + \\
+\,\mathrm{e}^{-\mathbf{A}t}\mathbf{a}(t)\,\mathrm{d}t &+ \mathrm{e}^{-\mathbf{A}t}\mathbf{b}(t)\,\mathrm{d}W(t).
\end{aligned}$$

We have

$$\mathrm{d}\left(\mathrm{e}^{-\mathbf{A}\,t}\cdot \mathbf{X}(t)\right) = \mathrm{e}^{-\mathbf{A}t}\left(\mathbf{a}(t)\,\mathrm{d}t + \mathbf{b}(t)\,\mathrm{d}W(t)\right).$$

Integrating the last equation we get

$$\mathrm{e}^{-\mathbf{A}\,t}\cdot\mathbf{X}(t) - \mathbf{X}(0) = \int_0^t \mathrm{e}^{-\mathbf{A}s}\,\mathbf{a}(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\mathbf{A}s}\,\mathbf{b}(s)\,\mathrm{d}W(s).$$

From this we can easily get the solution

$$\mathbf{X}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{X}(0) + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\,\mathbf{a}(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\,\mathbf{b}(s)\,\mathrm{d}W(s). \tag{9}$$

The solution $\mathbf{X}(t)$ is a random process and for its expectation we have for every $t > 0$

$$E[\mathbf{X}(t)] = \mathrm{e}^{\mathbf{A}\,t}\cdot E[\mathbf{X}(0)] + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\,\mathbf{a}(s)\,\mathrm{d}s, \tag{10}$$

while the expectation of the Itô integral is zero. We can see, that for constant initial value $\mathbf{X}(0)$, which is usually the case, the expectation of the stochastic solution coincides with the deterministic solution of the equation (8). By the deterministic solution we mean the analytic solution of the ordinary differential equation

$$\mathrm{d}\mathbf{X}(t) = \big(\mathbf{A}\,\mathbf{X}(t) + \mathbf{a}(t)\big)\,\mathrm{d}t. \tag{11}$$

E. Solution with Multiplicative Noise

We will solve the linear stochastic equation :

$$\mathrm{d}\mathbf{X}(t) = \big(\mathbf{A}\,\mathbf{X}(t) + \mathbf{a}(t)\big)\,\mathrm{d}t + \mathbf{B}\,\mathbf{X}(t)\,\mathrm{d}W(t), \tag{12}$$

where $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices and $\mathbf{a}(t) : \langle 0,T\rangle \to \mathbb{R}^2$ is a vector function.

We define a function $g(t,x_1,x_2,y) : \langle 0,T\rangle \times \mathbb{R}^3 \to \mathbb{R}^2$ by

$$g(t,x_1,x_2,y) = \mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}y}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{13}$$

and compute the derivative of $g(t,\mathbf{X}(t),W(t))$ by the Itô formula.

$$\begin{aligned}
&\mathrm{d}\Big(\mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}\mathbf{X}(t)\Big) = \\
&= \mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}\left(\frac{1}{2}\mathbf{B}^2 - \mathbf{A}\right)\mathbf{X}(t)\,\mathrm{d}t + \\
&+\mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}(-\mathbf{B})\mathbf{X}(t)\,\mathrm{d}W(t) + \\
&+\mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}\,\mathrm{d}\mathbf{X}(t) + \\
&+\frac{1}{2}\mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}(-\mathbf{B})(-\mathbf{B})\,\mathrm{d}^2W(t) + \\
&+\frac{1}{2}\mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}(-\mathbf{B})\big(\,\mathrm{d}\mathbf{X}(t)\cdot\,\mathrm{d}W(t)\big) + \\
&+\frac{1}{2}\mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}(-\mathbf{B})\big(\,\mathrm{d}W(t)\cdot\,\mathrm{d}\mathbf{X}(t)\big) = \\
&= \mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}\Big(\frac{1}{2}\mathbf{B}^2\mathbf{X}(t)\,\mathrm{d}t - \mathbf{A}\mathbf{X}(t)\,\mathrm{d}t - \mathbf{B}\mathbf{X}(t)\,\mathrm{d}W(t) + \\
&+\mathbf{A}\mathbf{X}(t)\,\mathrm{d}t + \mathbf{a}(t)\,\mathrm{d}t + \mathbf{B}\mathbf{X}(t)\,\mathrm{d}W(t) + \frac{1}{2}\mathbf{B}^2\mathbf{X}(t)\,\mathrm{d}t + \\
&+\frac{1}{2}(-\mathbf{B})\mathbf{B}\mathbf{X}(t)\,\mathrm{d}t + \frac{1}{2}(-\mathbf{B})\mathbf{B}\mathbf{X}(t)\,\mathrm{d}t\Big) = \\
&= \mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}\mathbf{a}(t)\,\mathrm{d}t.
\end{aligned}$$

We get

$$\mathrm{d}\Big(\mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}\mathbf{X}(t)\Big) = \mathrm{e}^{(\frac{1}{2}\mathbf{B}^2-\mathbf{A})t-\mathbf{B}W(t)}\mathbf{a}(t)\,\mathrm{d}t. \tag{14}$$

After the integration and some computation as in section D, we get the solution

$$\mathbf{X}(t) = \mathrm{e}^{(\mathbf{A}-\frac{1}{2}\mathbf{B}^2)t+\mathbf{B}W(t)}\mathbf{X}(0) +$$

$$\int_0^t \mathrm{e}^{(\mathbf{A}-\frac{1}{2}\mathbf{B}^2)(t-s)+\mathbf{B}(W(t)-W(s))}\,\mathbf{a}(s)\,\mathrm{d}s. \tag{15}$$

If $\mathbf{X}(0)$ is constant, the expectation of the solution, $E[\mathbf{X}(t)]$ for $t \in \langle 0,T\rangle$ is the unique solution of the ordinary differential equation, (see [1])

$$\mathrm{d}E[\mathbf{X}(t)] = \big(\mathbf{A}\,E[\mathbf{X}(t)] + \mathbf{a}(t)\big)\,\mathrm{d}t. \tag{16}$$

III. RLC Electric Circuit

A. Deterministic Model

Let $Q(t)$ be the charge at time t at a fixed point in an electric circuit, and let L be the inductance, R the resistance and $U(t)$ the potential source at time t. The charge $Q(t)$ satisfies the differential equation

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = U(t), \tag{17}$$

with initial conditions $Q(0) = Q_0,\ Q'(0) = I_0$. We can transform this equation by introducing the vector

$$\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} Q(t) \\ Q'(t) \end{pmatrix}$$

to the system

$$\begin{aligned} X_1' &= X_2 \\ X_2' &= -\frac{1}{CL}X_1 - \frac{R}{L}X_2 + \frac{U(t)}{L}. \end{aligned} \tag{18}$$

This in matrix notation gives

$$\frac{\mathrm{d}\mathbf{X}(t)}{\mathrm{d}t} = \mathbf{A}\cdot\mathbf{X}(t) + \mathbf{a}(t), \quad \mathbf{X}(0) = \mathbf{X}_0 \tag{19}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}, \ \mathbf{a}(t) = \begin{pmatrix} 0 \\ \frac{U(t)}{L} \end{pmatrix}, \ \mathbf{X}_0 = \begin{pmatrix} Q_0 \\ I_0 \end{pmatrix}.$$

The equation (19) has the analytic solution

$$\mathbf{X}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{X}_0 + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{a}(s)\ \mathrm{d}s. \tag{20}$$

B. RLC Circuit with Stochastic Source

We will consider the source influenced by random effects. Instead of $U(t)$ we consider the non deterministic version of this function:

$$U^*(t) = U(t) + \text{“noise”}. \tag{21}$$

To be able to substitute this into the equation (17) we have to describe mathematically the “noise”. It is reasonable to look at it as a stochastic process called the “white noise process”, denoted by $\xi(t)$. We get the following equation (α is a constant)

$$\frac{\mathrm{d}\mathbf{X}(t)}{\mathrm{d}t} = \mathbf{A}\cdot\mathbf{X}(t) + \mathbf{a}(t) + \begin{pmatrix} 0 \\ \frac{\alpha}{L}\xi(t) \end{pmatrix}. \tag{22}$$

We multiply (22) by $\mathrm{d}t$ and then replace $\xi(t)\,\mathrm{d}t$ by $\mathrm{d}W(t)$; $W(t)$ is the Wiener process. Formally the “white noise” is the time derivative of the Wiener process $W(t)$. We get

$$\mathrm{d}\mathbf{X}(t) = \Big(\mathbf{A}\cdot\mathbf{X}(t) + \mathbf{a}(t)\Big)\ \mathrm{d}t + \begin{pmatrix} 0 \\ \frac{\alpha}{L}\ \mathrm{d}W(t) \end{pmatrix}.$$

We got the equation of the form(8)

$$\mathrm{d}\mathbf{X}(t) = \Big(\mathbf{A}\cdot\mathbf{X}(t) + \mathbf{a}(t)\Big)\ \mathrm{d}t + \mathbf{b}\ \mathrm{d}W(t), \text{ where } \mathbf{b} = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix},$$

with the solution

$$\mathbf{X}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{X}(0) + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\ \mathbf{a}(s)\ \mathrm{d}s + \mathbf{b}\int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\ \mathrm{d}W(s). \tag{23}$$

and the expectation equal (20).

C. RLC Circuit with Stochastic Resistance

We consider now the resistance influenced by random effects. Instead of R we have:

$$R^* = R + \text{“noise”} = R + \alpha\xi(t), \tag{24}$$

where $\xi(t)$ denotes the “white noise process”, α is a constant. We put this to the form(17) and rewrite the second order equation as a system of two equations, then multiply both equations by $\mathrm{d}t$ and replace $\xi(t)\,\mathrm{d}t$ by $\mathrm{d}W(t)$. We get the stochastic vector equation (12)

$$\mathrm{d}\mathbf{X}(t) = \big(\mathbf{A}\,\mathbf{X}(t) + \mathbf{a}(t)\big)\ \mathrm{d}t + \mathbf{B}\,\mathbf{X}(t)\ \mathrm{d}W(t),$$

where $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\alpha}{L} \end{pmatrix}$, $\mathbf{A}$ and $\mathbf{a}(t)$ are as in (19), with the solution (15).

The expectation of $\mathbf{X}(t)$ solves the equation (19), so the expectation $E[\mathbf{X}(t)]$ is equal to the deterministic solution (20).

D. SDE Numerical Methods Applied

For practical use of stochastic differential equations we have to simulate the stochastic solution by numerical techniques. The simplest numerical scheme, the stochastic Euler scheme, is based on the Euler numerical scheme for ordinary differential equations.

Let us consider an equidistant discretisation of the time interval $\langle 0, T\rangle$, namely

$$t_k = kh, \text{ where } h = \frac{T}{K} = t_{k+1} - t_k = \int_{t_k}^{t_{k+1}} \mathrm{d}t,$$

$K \in \mathbb{N},\ k = 0, \dots, K-1$ and the corresponding discretisation of the $j-$th component of the Wiener process,

$$\Delta W_k^j = W^j(t_{k+1}) - W^j(t_k) = \int_{t_k}^{t_{k+1}} \mathrm{d}W^j(s).$$

To be able to apply a stochastic numerical scheme, first we have to generate, for all j, the random increments of W^j as independent Gauss random variables with mean $E[\Delta W_k^j] = 0$ and $E[(\Delta W_k^j)^2] = h$.

The explicite Euler scheme for the $i-$th component of the N dimensional stochastic differential equation (1) has the form

$$X_{k+1}^i = X_k^i + A^i(t_k, \mathbf{X}_k)h + \sum_{j=1}^{M} B^{i,j}(t_k, \mathbf{X}_k)\Delta W_k^j. \tag{25}$$

For measuring the accuracy of a numerical solution to an SDE we use the strong order of convergence. The Euler scheme converges with strong order $\frac{1}{2}$ (see [3]).

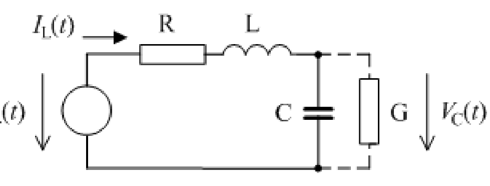

Fig. 1. RLC(G) circuit with unit-step voltage source.

IV. Examples

A. Example 1

First we consider the RLC circuit, see Fig. 1, excited from a unit-step voltage source influenced by a noise,

$$\underline{1}^*(t) = \underline{1}(t) + \alpha\xi(t),$$

with remaining circuit parameters deterministic.

The values of all the parameters are unit which corresponds to an underdamped behavior of the circuit. Hereafter we consider the capacitor voltage $V_C(t) = Q(t)/C$ and the inductor current $I_L(t) = Q'(t)$ as state variables in the circuit instead of the charge-based notations in (17)-(19), which will be useful for further considerations. Also zero initial conditions are considered in this example.

Current and voltage responses, namely their 100 realizations, including their sample means accompanied by 99% confidence intervals highlighted, are presented in Fig. 2. Deterministic solutions based on (20) lead to formulae

$$i_L(t) = \frac{1}{\omega L} \mathrm{e}^{-\beta t} \sin \omega t \tag{26}$$

$$v_C(t) = 1 - \mathrm{e}^{-\beta t} \left(\frac{\beta}{\omega} \sin \omega t + \cos \omega t \right) \tag{27}$$

with

$$\beta = \frac{R}{2L} \quad \text{and} \quad \omega = \sqrt{\frac{1}{LC} - \beta^2}.$$

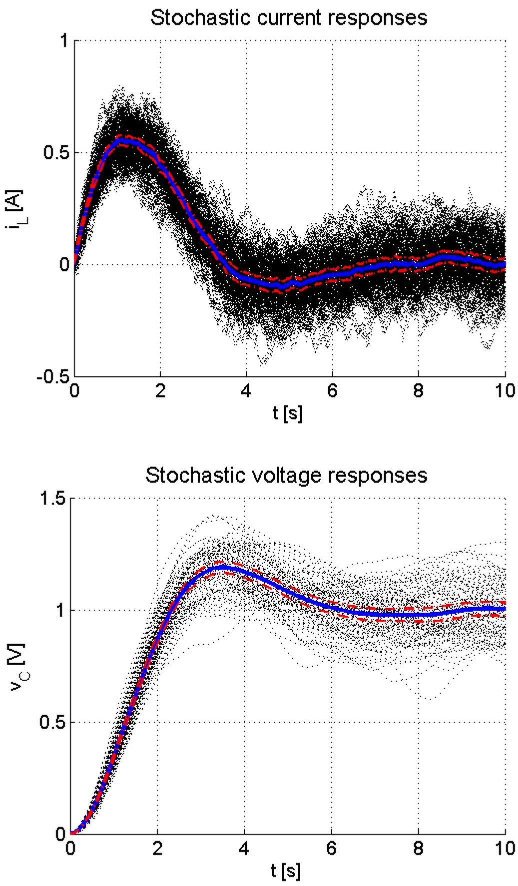

Fig. 2. RLC circuit stochastic responses ($\alpha = 0.15$, noisy source).

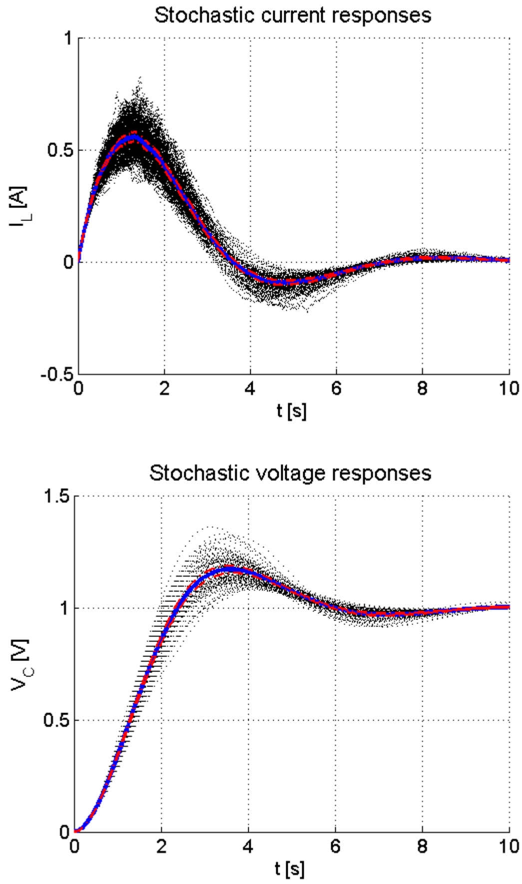

Fig. 3. RLC circuit stochastic responses ($\alpha = 0.15$, noisy resistor).

It can easily be confirmed correspondences between the sample means in Fig. 2 and waveforms based on (26) and (27).

Some dissimilarities in the above stochastic processes can be observed, "smoother" character of the voltages is likely caused by a filtering effect of the low-pass filter formed by this network.

B. Example 2

The second example considers the RLC circuit excited from a deterministic unit-step voltage source $\underline{1}(t)$, with L and C deterministic as well, but with R influenced by a noise, i.e. its resistance is equal $R^*(t) = R + \alpha\xi(t)$.

The current and voltage responses (their 100 realizations), including their sample means with 99% confidence intervals highlighted, are depicted in Fig. 3. The voltage stochastic waveforms look again "smoother" than the current ones, even more when comparing with Fig. 2, and with less dispersions for the same noise intensity factor α.

C. Example 3

The third example considers an RLCG circuit, or the RLC circuit terminated by a resistive load . The conductance G can also model nonideal properties of a real electrical condenser, namely its nonzero leakage. In this case the matrix $\mathbf{A}$ defined in (19) changes as

$$\mathbf{A} = \begin{pmatrix} -\frac{G}{C} & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix},$$

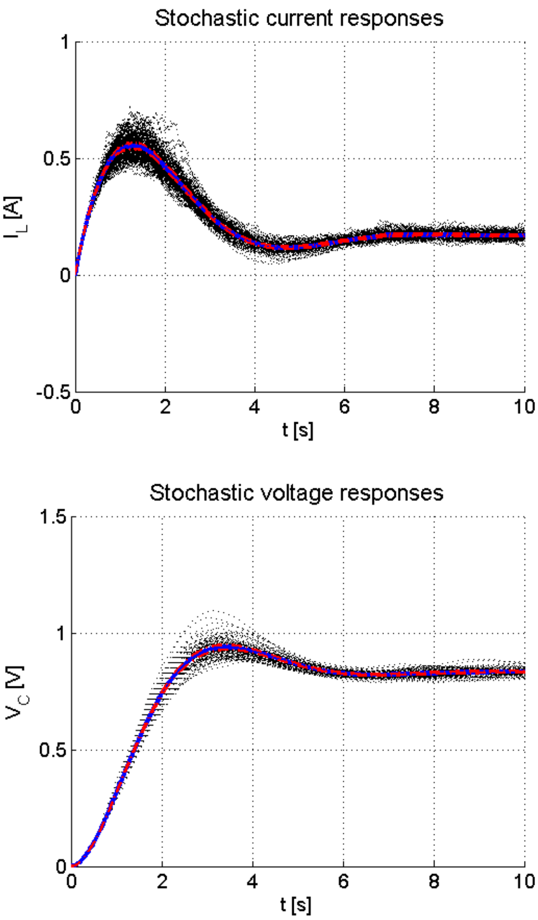

Fig. 4. RLCG circuit stochastic responses ($\alpha = 0.15$, noisy resistor).

remaining matrices keep their forms. When a conductance is chosen e.g. as $G = 0.2$S, the circuit stays underdamped, with analytical solutions presented e.g. in [14]. Numerical solutions for stochastic exciting voltage source are, from qualitative viewpoint, very similar to those in Fig. 2. In case of stochastic resistance R, however, the results are presented in Fig. 4. Here it is obvious that due to a permanent current flowing through the resistor R, stochastic responses do not tend to be damped down to zero, unlike the RLC circuit in Fig. 3.

V. Uniform Transmission Line Modelling

Here a uniform transmission line (TL) lumped-parameter model will be considered and its responses simulated, see Fig. 5. If we denote l as a TL length and L_0, R_0, C_0 and G_0 as its per-unit-length parameters the lumped parameters of the model are defined by $L_d = L_0 l/m, R_d = R_0 l/m, C_d = C_0 l/m$, and $G_d = G_0 l/m$, where m is a number of Π sections in cascade. For the Thévenin resistances of terminal circuits supposed as nonzero, the terminal currents are given by $i_S = (v_S - v_1)/R_S$, and $i_L = v_L/R_L = v_{m+1}/R_L$. The asterisks at an exciting voltage, v_S^*, and series resistances, R_d^*, mean that respective quantity is considered as stochastically varying.

A. Deterministic Model

As is shown e.g. in [16] a uniform TL m-sectional model can be described by a state-variable method leading to a vector ordinary differential equation

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{Ax}(t) + \mathbf{Bu}(t) \tag{28}$$

with a well-known analytical solution

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{Bu}(s)\,\mathrm{d}s \tag{29}$$

Formal similarities with (19) and (20) suggest us to proceed with stochastic solutions by similar way, see next chapters. The individual terms in (28) will be formulated as follows. The matrices $\mathbf{A} = -\mathbf{M}^{-1}(\mathbf{H} + \mathbf{P})$, and $\mathbf{B} = \mathbf{M}^{-1}\mathbf{P}$, where

$$\mathbf{M} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{pmatrix}, \ \mathbf{H} = \begin{pmatrix} \mathbf{G} & \mathbf{E} \\ \mathbf{-E^T} & \mathbf{R} \end{pmatrix}, \ \mathbf{P} = \begin{pmatrix} \mathbf{Y_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{30}$$

with $\mathbf{C} = C_d\mathbf{I}_{m+1}, \mathbf{L} = L_d\mathbf{I}_m, \mathbf{G} = G_d\mathbf{I}_{m+1}$, and $\mathbf{R} = R_d\mathbf{I}_m$, as diagonal matrices (there are exceptions in $\mathbf{C}$ and $\mathbf{G}$ related to boundary elements, when $C_d/2$ and $G_d/2$ are used), $\mathbf{I}$ identity matrices of indexed orders, and $\mathbf{0}$ corresponding zero matrix. The $\mathbf{E}$ is a special $(m+1) \times m$ bidiagonal matrix containing ± 1 and zeros, see [16] for more details, and the $(m+1)$-order matrix $\mathbf{Y}_E = \mathrm{diag}(G_S, 0, ..., 0, G_L)$ is dependent on the external circuits, where $G_S = R_S^{-1}$ and $G_L = R_L^{-1}$. The column vector $\mathbf{x}(t) = [\mathbf{v}_C^T(t), \mathbf{i}_L^T(t)]^T$ consists of the state variables required, namely of subvectors $\mathbf{v}_C(t)$ and $\mathbf{i}_L(t)$ holding $m+1$ capacitor voltages and m inductor currents, respectively. Finally, the column vector $\mathbf{u}(t) = [\mathbf{v}_E^T(t), \mathbf{0}]^T$ contains the $(m+1)$-order vector $\mathbf{v}_E(t) = [v_S(t), 0, ..., 0]^T$ acting as an excitation term.

B. Model with Stochastic Excitation

Here, instead of $v_S(t)$, a non-deterministic version is used

$$v_S^*(t) = v_S(t) + \alpha\xi(t). \tag{31}$$

while remaining parameters are deterministic.

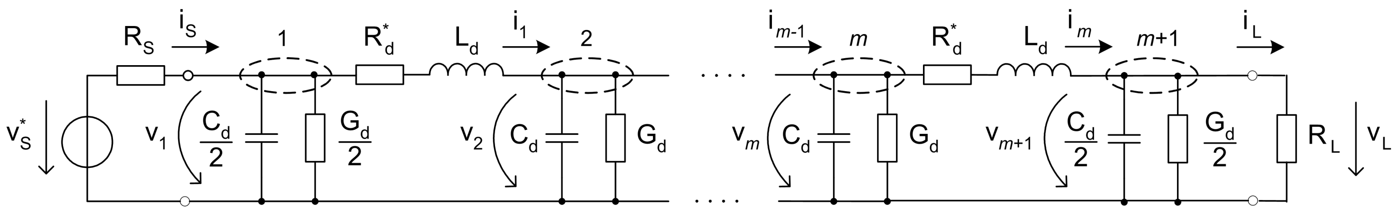

Fig. 5. Uniform transmission line m-sectional lumped-parameter model.

A noise term $\xi(t)$ is a stochastic process, namely a white noise, and with α a constant, expressing its intensity. After substitution (31) into (28) and doing arrangements keeping the SDE theory above we get a vector linear SDE with an additive noise

$$\mathrm{d}\mathbf{X}(t) = (\mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{u}(t))\ \mathrm{d}t + \mathbf{b}\,\mathrm{d}W(t) \tag{32}$$

with $\mathbf{b} = \mathbf{B}[\boldsymbol{\alpha}_E^T, \mathbf{0}]^T$, and $\alpha_E = [\alpha, 0, \ldots, 0]^T$ as a noise intensity vector of the order $m+1$. The stochastic solution $\mathbf{X}(t)$ in (32) is marked by a capital letter to distinguish it from that deterministic in (29). After some modification of the theory from section two, considering the multidimensional Itô formula [1], [2], the formal analytical stochastic solution is

$$\mathbf{X}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{X}_0 + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\,\mathbf{B}\mathbf{u}(s)\ \mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{b}\ \mathrm{d}W(s), \tag{33}$$

with the right term called as the Itô integral. The $\mathbf{X}(t)$ is a random process and for its expectation we have for $t > 0$

$$E[\mathbf{X}(t)] = \mathrm{e}^{\mathbf{A}t}E[\mathbf{X}_0] + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\,\mathbf{B}\mathbf{u}(s)\ \mathrm{d}s, \tag{34}$$

when the expectation of the Itô integral is zero. It is evident that for an initial value $\mathbf{X}_0$ being constant, this expectation coincides with the deterministic solution (29).

C. Model with Stochastic Resistances

Herein the responses to a deterministic excitation $v_S(t)$ but under all m resistances R_d in the TL model influenced by random effects are considered. Instead of the original R_d, their non-deterministic versions are used as

$$R^*_{dk}(t) = R_d + \alpha_k \xi_k(t). \tag{35}$$

$k = 1, \ldots, m$, where noise terms $\xi_k(t)$ are again considered as "white noise processes", and α_k as their intensities. Notice that the series resistances connected to different nodes are generally affected by different noise terms like it can occur on a real TL. After substitution (35) into (28), and keeping the SDE theory procedures, we get

$$\mathrm{d}\mathbf{X}(t) = (\mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{u}(t))\ \mathrm{d}t + \mathbf{D}_W(t)\mathbf{X}(t) \tag{36}$$

with

$$\mathbf{D}_W = -\mathbf{M}^{-1}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha_R}\ \mathbf{dW_R}(\boldsymbol{t}) \end{pmatrix}, \tag{37}$$

where the matrices $\alpha_R = \mathrm{diag}(\alpha_1, ..., \alpha_m)$ and $\mathrm{d}\mathbf{W}_R(t) = \mathrm{diag}(\ \mathrm{d}W_1(t), ...,\ \mathrm{d}W_m(t))$. This is a general vector linear SDE with a multiplicative noise, which is represented by a m-dimensional Wiener process. The solution in general doesn't have the normal distribution for X_0 constant see [1], but the expectation of the solution (36) is given again by (34), its exact analytical solution can be obtained by generalizing results shown e.g. in [17].

D. SDE Numerical Method Applied

As computer simulations revealed, the explicit Euler scheme (25) satisfies only for relatively low orders of electric models under consideration to get stable solutions. That is why, implicit Euler schemes will be used for all the practical simulations of the TL model.Then, based on [3] and considering (32) and (36), implicit stochastic Euler schemes can be formulated respectively as

$$\mathbf{X}^{n+1} = (\mathbf{I} - \mathbf{A}h)^{-1}\left(\mathbf{X}^n + \mathbf{B}\mathbf{u}^{n+1}h + \mathbf{b}\Delta W^n\right) \tag{38}$$

for the SDE with an additive noise, and

$$\mathbf{X}^{n+1} = (\mathbf{I} - \mathbf{A}h)^{-1}\left(\mathbf{X}^n + \mathbf{B}\mathbf{u}^{n+1}h + \mathbf{D}^n_W\mathbf{X}^n\right) \tag{39}$$

for the SDE with a multiplicative noise, where $\mathbf{I}$ is the identity matrix, being consistent with the Itô stochastic calculus.

E. Examples of TL Model Responses

Let us now consider a uniform TL with per-unit-length parameters $R_0 = 0.1\Omega$/m, $L_0 = 494.6$nH/m,$G_0 = 0.1$S/m, and $C_0 = 62.8$pF/m [15]. The TL's length $l = 0.3$m, the terminal resistances $R_S = 10\Omega$ and $R_L = 1\mathrm{k}\Omega$, the drive source is a sine-squared impuls, $v_S(t) = \sin^2(\pi t/2 \cdot 10^{-9})$ if $0 \le t \le 2 \cdot 10^{-9}$s, and $v_S(t) = 0$ otherwise. To demonstrate overall state on the TL the deterministic solutions for voltage and current distributions are depicted in Fig. 6, computed through a continuous model in a Laplace domain to be more accurate for a comparison, and by using a numerical inversion of Laplace transforms [15].

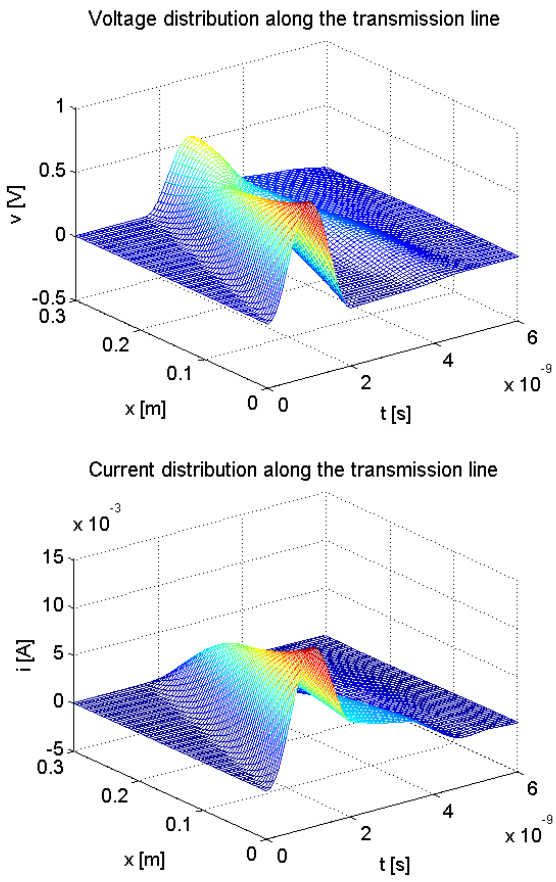

Fig. 6. Voltage and current distributions along the uniform TL.

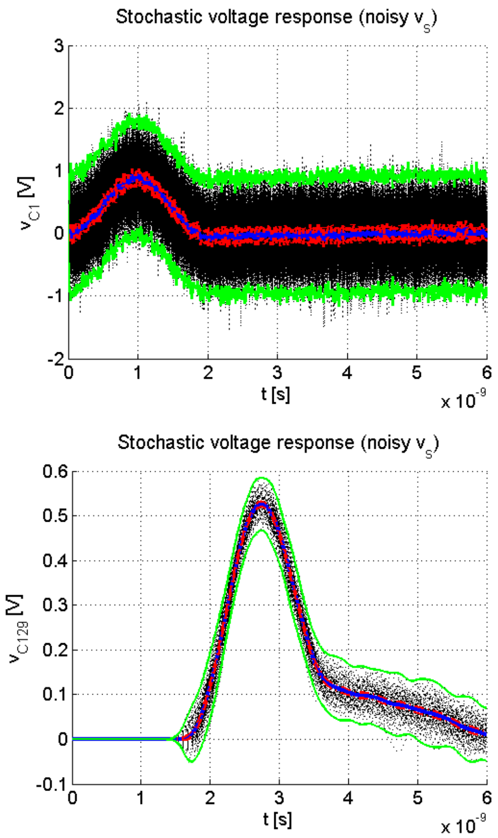

Fig. 7. Stochastic voltage responses for noisy excitation voltage.

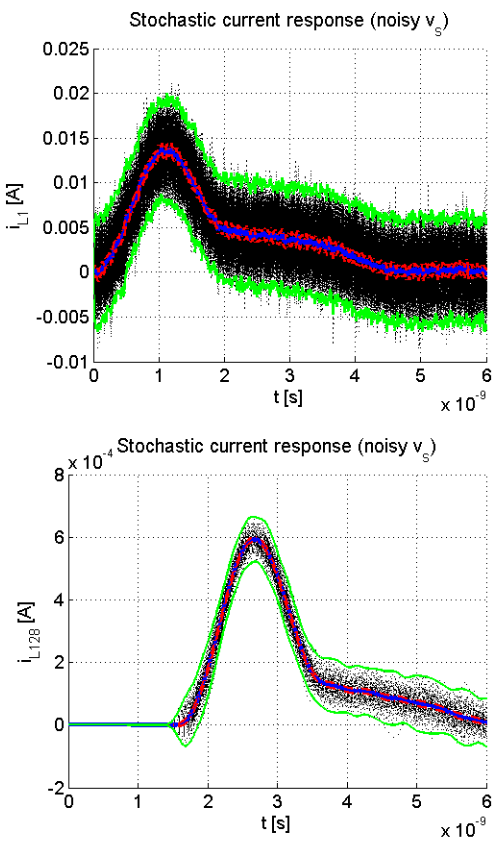

Fig. 8. Stochastic current responses for noisy excitation voltage.

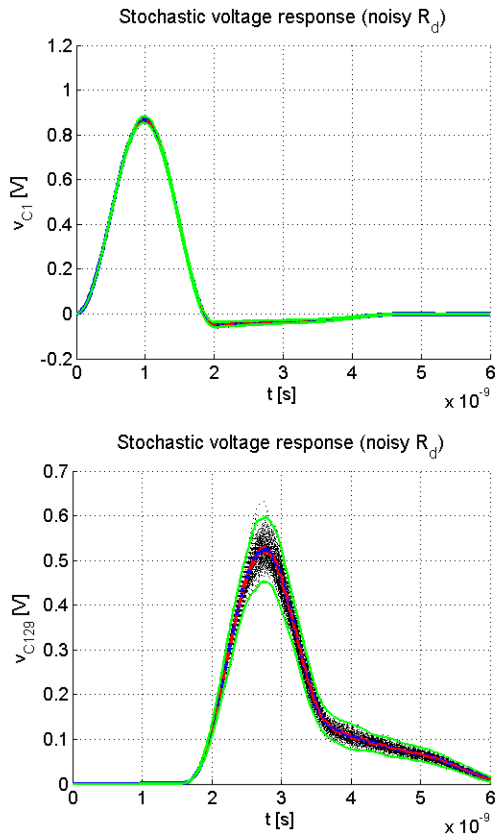

Fig. 9. Stochastic voltage responses for noisy series resistances.

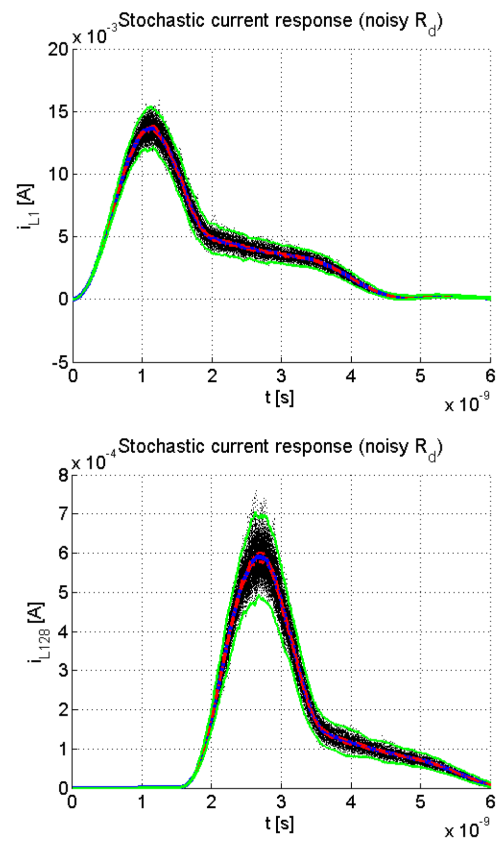

Fig. 10. Stochastic current responses for noisy series resistances.

The Fig. 7 and 8 show a case of additive noises, Fig. 9 and 10 then a case of multiplicative ones. The computation was performed for $m = 128$ sections, which complies, with some margin, with rule of thumb the TL section delay is less then roughly a tenth of the rise/fall time of the impuls propagated ($m > 100$ would be good in our case, a choice 128 simplifies the usage of another comparative method, a 2D LT in conjunction with an FFT-based 2D-NILT algorithm).

The 99% confidence intervals (dashed red lines), marking uncertainities in determining the sample mean values (solid blue lines) are highlighted, together with the confidence intervals for individual samples (solid green lines), marking a measure of stochastic trajectories dispersion. In case of the additive noise the resultant stochastic processes keep their Gaussian distribution, and a student-t distribution can be directly applied. In case of the multiplicative noise, however, this is not valid, and the sample statistics are calculated via dividing the samples set into a number of batches while utilizing a central limit theorem, see [3] for more details. The noise intensities were chosen as $\alpha = 10^{-6}$ in case of the noisy source v_S^*, and $\alpha_k = 10^{-5}, \forall k$, in case of the noisy series resistances R_d^*. Distinct noise intensities for different noisy elements, which result in comparable dispersions of stochastic processes, can help to evaluate components of the TL model with significant noise effects. It should be noticed that stochastic processes $\mathrm{d}W_k(t)$, needed for all numerical solutions, are generated by the MATLAB® function for normally distributed random numbers.

VI. Conclusion

Real implementations of the interconnects in high-speed electronic systems are being influenced by a number of physical effects [13]. The stochastic differential equations approach can therefore be an interesting alternative to other probabilistic approaches, how to process such the random effects. We can mention at least [18], where a stochastic Galerkin method for the solution of stochastic telegrapher's equations is used, [19], where a polynomial-chaos expansion method is adopted, or [20], where a method based on a classical Monte Carlo analysis is considered. In mentioned cases, however, the solutions are performed in the frequency domain, in contrast to our work, where the responses in the time domain are directly treated. Especially at the solution of signal integrity issues an overall dispersion in the time domain can help to evaluate whether digital signals delivered are treatable by receivers. However, the application of the SDE theory in the TLs modelling in the frequency domain is also enabled.

We will focus our further research on a few directions. From the point of view of theoretical studies the remaining RLCG circuit elements will be taken into account to produce stochastic effects, and also their simultaneous actions will be investigated. A SDEs approach will again be used to model higher-order systems, especially those with multiconductor transmission lines (MTL) [15], [21]. Finally, other higher-order numerical techniques consistent with the Itô stochastic calculus will further be considered to ensure even better stability and accuracy of the solutions.

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